Displacement of an Extended Body by Metric Deformation

The Geometric Displacement Theorem

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Abstract

Let $(M, \bar{g})$ be a compact Riemannian manifold without boundary and $\Omega \subset M$ an extended body with smooth mass density $\rho > 0$, initially at rest. When the metric evolves along a smooth family $g(t)$ with $g(0) = \bar{g}$, the Karcher center of mass $p(t)$ of Ω is displaced — without any applied force and without initial velocity. We prove the bound

$$d_{\bar{g}}(p(T),\, p(0)) \;\leq\; C(\Omega, \rho, \bar{g}) \cdot \mathcal{D}_\Omega,$$

where $\mathcal{D}_\Omega := \int_0^T \int_\Omega \big(\|\dot{g}\|_{g(t)} + \|\nabla_{g(t)} \dot{g}\|_{g(t)} \big)\, dV_{g(t)}\, dt$ is the deformation functional restricted to Ω. In particular, $p(t) \equiv p(0)$ if and only if g is static on Ω. This is the *Geometric Displacement Theorem*, a strict Riemannian generalization of Newton's first law: an extended body at rest remains stationary if and only if the ambient space is locally static.

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Introduction

Newton's first law states that a body at rest remains at rest in the absence of external forces — in a fixed Euclidean space.

When the space itself deforms, this statement requires revision. A metric deformation modifies distances, curves geodesics, and displaces the center of mass of an extended body even when each material point is locally at rest. This phenomenon has no counterpart in classical mechanics, where space is assumed to be rigid.

The present note establishes a quantitative version of this observation. The displacement of the Karcher center of mass is controlled, in a precise and computable way, by the *deformation functional* $\mathcal{D}_\Omega$ — an integral of the metric velocity over Ω.

The proof combines three classical ingredients: the Karcher mean [2, 3], the geodesic energy variation formula [6], and the implicit function theorem in infinite dimensions. Their combination yields a result that, to our knowledge, has not been stated in this form in the literature.

Setup and Definitions

Let $(M, \bar{g})$ be a smooth compact Riemannian manifold without boundary, of dimension $n \geq 2$. Denote by $d_{\bar{g}}$ the Riemannian distance and by $dV_{\bar{g}} = \sqrt{\det \bar{g}}\, d^n x$ the Riemannian volume element.

Definition 1 (Metric family)**.** *A* smooth metric family *on $[0,T]$ is a C^∞ map $t \mapsto g(t)$ with values in the space of Riemannian metrics on M, satisfying $g(0) = \bar{g}$. Set $\dot{g}_{ij}(x,t) := \partial_t g_{ij}(x,t)$ and $h(t) := g(t) - \bar{g}$ (the deformation tensor). We assume $\|h\|_{C^2(M)} \leq \delta$ for some small $\delta > 0$ (small-deformation regime).*

Definition 2 (Deformation functional)**.** *Let $\Omega \subset M$ be a relatively compact open set. The* deformation functional *restricted to Ω is:*

$$\mathcal{D}_\Omega := \int_0^T \int_\Omega \Big(\|\dot{g}(x,t)\|_{g(t)} + \|\nabla_{g(t)} \dot{g}(x,t)\|_{g(t)} \Big)\, dV_{g(t)}(x)\, dt, \tag{1}$$

where $\|\dot{g}\|^2_{g(t)} = g^{ik}(t)\, g^{jl}(t)\, \dot{g}_{ij}\, \dot{g}_{kl}$ *is the Riemannian Frobenius norm with respect to the* current *metric* $g(t)$*, and* $\nabla_{g(t)}$ *is its Levi-Civita connection. Since* $\|h\|_{C^2} \leq \delta$ *is small, the norms with respect to* $g(t)$ *and* $\bar{g}$ *are equivalent up to a factor* $(1 + O(\delta))$*, so the choice of metric for the norm does not affect the validity of the bound* (4)*. Using the current metric* $g(t)$ *is the natural choice: it makes* $\mathcal{D}_\Omega$ *intrinsic to the evolving geometry, and yields clean expressions in applications such as the Ricci flow (Section 6).*

Definition 3 (Karcher center of mass)**.** *Let* $\Omega \subset M$ *be a relatively compact open set contained in a geodesic ball of radius* $r < \pi/(2\sqrt{\kappa})$*, where* $\kappa = \sup_M K_{\bar{g}} \geq 0$ *is the maximum sectional curvature of* $\bar{g}$*. Let* $\rho \in C^\infty(\Omega)$ *be a mass density,* $\rho > 0$*.*

For δ *small enough, the deformed metrics* $g(t)$ *have sectional curvature bounded by* $\kappa' \leq \kappa + C\delta$*, and* Ω *remains inside a geodesically convex ball for all* $g(t)$ *[6]. Consequently, the functional* $F(\cdot, t)$ *defined below is strictly convex and its Hessian is uniformly invertible over* $[0, T]$*.*

The Karcher center of mass *of* (Ω, ρ) *in the metric* $g(t)$ *is the unique point* $p(t) \in M$ *minimizing:*

$$F(q, t) \;:=\; \int_\Omega \rho(x)\, d_{g(t)}(q, x)^2\, dV_{g(t)}(x). \tag{2}$$

Existence and uniqueness of $p(t)$ *for* δ *small enough are guaranteed by [3].*

Remark 1. *The smallness assumption* $\|h\|_{C^2} \leq \delta$ *serves two purposes: it ensures uniform convexity of* $F(\cdot, t)$ *for all* $g(t)$*, and it controls the* $O(\delta^2)$ *remainder terms in the proof. The constant* C *in* (4) *does not depend on* δ *in this regime.*

Key Lemma

Lemma 1 (Variation of squared distance)**.** *Let* $p, x \in M$ *and* $\gamma : [0, 1] \to M$ *be the minimizing geodesic of* $\bar{g}$ *from* $\gamma(0) = p$ *to* $\gamma(1) = x$*, parametrized at constant speed. For* $g(t) = \bar{g} + t\, h + O(t^2)$*:*

$$\frac{d}{dt}\big[d_{g(t)}(p, x)^2\big]\Big|_{t=0} \;=\; \int_0^1 h\big(\gamma'(s),\, \gamma'(s)\big)\, ds. \tag{3}$$

Proof. The energy of γ in the metric $g(t)$ is:

$$E_{g(t)}[\gamma] = \int_0^1 g(t)\big(\gamma'(s), \gamma'(s)\big)\, ds.$$

For the minimizing geodesic of $\bar{g}$, we have $d_{\bar{g}}(p, x)^2 = E_{\bar{g}}[\gamma]$. Varying $E_{g(t)}[\gamma]$ with respect to g alone (the curve γ is held fixed as the $\bar{g}$-geodesic) gives:

$$\frac{d}{dt}\big[E_{g(t)}[\gamma]\big]\Big|_{t=0} = \int_0^1 h\big(\gamma'(s), \gamma'(s)\big)\, ds.$$

It remains to justify that $\frac{d}{dt}[d^2_{g(t)}] = \frac{d}{dt}[E_{g(t)}[\gamma]]$ to first order. For δ small, the minimizing geodesic of $g(t)$ is a C^1 perturbation of γ [6]. By the variational principle, the energy variation along this perturbed geodesic is $O(\delta^2)$ (it is a critical point). Therefore:

$$\frac{d}{dt}\big[d_{g(t)}(p, x)^2\big]\Big|_{t=0} = \int_0^1 h(\gamma', \gamma')\, ds + O(\delta^2),$$

which establishes (3) to first order. □ □

The Geometric Displacement Theorem

Theorem 1 (Geometric Displacement Theorem)**.** *Under the assumptions of Definition 3, there exists a constant $C = C(\Omega, \rho, \bar{g}) > 0$, independent of $g(t)$, such that:*

$$\boxed{d_{\bar{g}}(p(T),\, p(0)) \;\leq\; C \cdot \mathcal{D}_\Omega.} \tag{4}$$

Proof. The proof proceeds in four steps.

Step 1 — Equation of motion of $p(t)$. The center of mass $p(t)$ satisfies the optimality condition $\operatorname{grad}_q F(p(t), t) = 0$, where:

$$\operatorname{grad}_q F(q,t) = -2 \int_\Omega \rho(x)\, \exp^{-1}_{q,g(t)}(x)\, dV_{g(t)}(x).$$

Differentiating in t via the implicit function theorem (applicable since $\operatorname{Hess}_q F(p(t),t)$ is invertible by convexity):

$$\operatorname{Hess}_q F(p(t),t) \cdot \frac{dp}{dt} = -\frac{d}{dt}\big[\operatorname{grad}_q F(q,t)\big]\Big|_{q=p(t)}. \tag{5}$$

Step 2 — Bound on the right-hand side. We decompose the derivative into two contributions — variation of the *distance* and variation of the *volume element*:

$$\frac{d}{dt}\big[\operatorname{grad}_q F\big] = \underbrace{-2 \int_\Omega \rho(x) \frac{d}{dt}\big[\exp^{-1}_{q,g(t)}(x)\big] dV_{\bar{g}}}_{\text{(I) distance variation}}$$

$$\underbrace{-\,2 \int_\Omega \rho(x)\, \exp^{-1}_{q,\bar{g}}(x) \cdot \tfrac{1}{2} \operatorname{tr}_{\bar{g}}(h)\, dV_{\bar{g}}}_{\text{(II) volume element variation}} + O(\delta^2),$$

where term (II) arises from the expansion $dV_{g(t)} = \big(1 + \frac{1}{2}\operatorname{tr}_{\bar{g}}(h) + O(\delta^2)\big) dV_{\bar{g}}$.
For term (I), the variation of the exponential map is controlled by the derivatives of h. By [6, Theorem 1.28], the differential of the exponential map varies C^1-continuously with the metric, giving:

$$\left|\frac{d}{dt}\big[\exp^{-1}_{q,g(t)}(x)\big]\right|_{\bar{g}} \leq C_1(\bar{g})\; \|h\|_{C^1(\Omega)}\,.$$

Term (II) is bounded by $C_1'(\bar{g})\, \|h\|_{C^0(\Omega)} \leq C_1'(\bar{g})\, \|h\|_{C^1(\Omega)}$. Integrating over Ω against ρ and writing $h = \dot{g}\, dt$:

$$\left|\frac{d}{dt}\big[\operatorname{grad}_q F\big]\right|_{\bar{g}} \leq C_2(\Omega, \rho, \bar{g})\; \|\dot{g}(t)\|_{C^1(\Omega)}\,. \tag{6}$$

Step 3 — Bound on $|dp/dt|$. The Hessian of F at $p(t)$ satisfies, by uniform convexity of Ω for all metrics $g(t)$ (Definition 3):

$$\operatorname{Hess}_q F(p(t),t) \geq \lambda(\Omega, \rho, \bar{g}) \cdot \mathrm{Id} > 0.$$

From (5) and (6):

$$\left|\frac{dp}{dt}\right|_{\bar{g}} \leq \frac{C_2}{\lambda}\; \|\dot{g}(t)\|_{C^1(\Omega)}\,. \tag{7}$$

Step 4 — Integration and link to $\mathcal{D}_\Omega$. Integrating (7) over $[0,T]$:

$$d_{\bar{g}}(p(T),p(0)) \leq \int_0^T \left|\frac{dp}{dt}\right|_{\bar{g}} dt \leq \frac{C_2}{\lambda}\int_0^T \|\dot{g}(t)\|_{C^1(\Omega)}\, dt.$$

Since $\dot{g} \in C^2(M\times[0,T])$ by assumption, a partition-of-unity argument on the compact set Ω yields a constant $C_3(\Omega,\bar{g})>0$ such that, for each t:

$$\|\dot{g}(t)\|_{C^1(\Omega)} \leq C_3 \int_\Omega \Big(\|\dot{g}(x,t)\|_{\bar{g}} + \|\nabla\dot{g}(x,t)\|_{\bar{g}}\Big) dV_{\bar{g}}(x). \tag{8}$$

Indeed, on each coordinate chart U_α of a finite cover of Ω with cut-off functions ψ_α, the sup-norm of $\psi_\alpha\dot{g}$ is controlled by its $W^{1,1}(U_\alpha)$-norm via a standard one-dimensional Sobolev embedding ($W^{1,1}(\mathbb{R}) \hookrightarrow L^\infty$) applied along each coordinate direction; the constant C_3 depends on the cover and the transition maps. Integrating (8) over $[0,T]$ gives:

$$\int_0^T \|\dot{g}(t)\|_{C^1(\Omega)}\, dt \leq C_3\, \mathcal{D}_\Omega.$$

Setting $C := C_2C_3/\lambda$ yields (4). □ □

Corollaries

Corollary 1 (Characterization of rest)**.** *Under the assumptions of Theorem 1:*

$$p(t)=p(0) \ \text{ for all } t\in[0,T] \iff g(t)=\bar{g} \ \text{ on } \Omega \text{ for almost every } t.$$

Proof. ($\Leftarrow$): If $g(t)=\bar{g}$ on Ω, then $\dot{g}=0$ on Ω, so $\mathcal{D}_\Omega=0$, and by (4), $d_{\bar{g}}(p(t),p(0))=0$ for all t.

($\Rightarrow$): If $p(t)=p(0)$ for all t, then $|dp/dt|=0$. By (7) and the positivity of λ, $\|\dot{g}(t)\|_{C^1(\Omega)}=0$ for almost every t, which implies $\dot{g}\equiv 0$ on Ω, i.e., $g(t)=\bar{g}$. □ □

Corollary 2 (Geometric Inertia Law)**.** *An extended body (Ω,ρ) initially at rest remains stationary if and only if the ambient space is static on Ω.*

This corollary is the Riemannian generalization of Newton's first law. In a deformable space, rest is not the default natural state — it is a property of the space itself. The functional $\mathcal{D}_\Omega$ measures precisely to what extent the deformation of space is responsible for the observed motion.

Application: Ricci Flow

The Ricci flow provides a canonical example of a smooth metric family to which Theorem 1 applies directly. Recall that the Ricci flow is the evolution equation

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\, R_{ij}(g(t)), \tag{9}$$

introduced by Hamilton [4]. Here R_{ij} denotes the Ricci tensor of $g(t)$.

Proposition 1 (Deformation functional along Ricci flow)**.** *Along the Ricci flow* (9)*, the deformation functional takes the form:*

$$\mathcal{D}_\Omega = 2\int_0^T \int_\Omega \Big(|\mathrm{Ric}(g(t))|_{g(t)} + |\nabla \mathrm{Ric}(g(t))|_{g(t)} \Big)\, dV_{g(t)}\, dt, \tag{10}$$

where $|\mathrm{Ric}|_g^2 = g^{ik} g^{jl} R_{ij} R_{kl}$ *is the Hilbert–Schmidt norm of the Ricci tensor.*

Proof. Since $\dot{g}_{ij} = -2R_{ij}(g(t))$, we have:

$$\|\dot{g}\|_{g(t)}^2 = g^{ik} g^{jl} \dot{g}_{ij} \dot{g}_{kl} = 4\, g^{ik} g^{jl} R_{ij} R_{kl} = 4\, |\mathrm{Ric}(g(t))|_{g(t)}^2,$$

so $\|\dot{g}\|_{g(t)} = 2|\mathrm{Ric}(g(t))|_{g(t)}$. Similarly, $\nabla \dot{g}_{ij} = -2\nabla R_{ij}$, giving $\|\nabla \dot{g}\|_{g(t)} = 2|\nabla \mathrm{Ric}(g(t))|_{g(t)}$. Substituting into (1) yields (10). □ □

Combined with Theorem 1, this gives:

$$d_{\bar{g}}(p(T), p(0)) \;\le\; 2C \int_0^T \int_\Omega \Big(|\mathrm{Ric}|_g + |\nabla \mathrm{Ric}|_g \Big)\, dV_g\, dt. \tag{11}$$

The displacement of the center of mass is thus controlled by the integral of the curvature over $\Omega \times [0, T]$.

Einstein manifolds. If $\bar{g}$ satisfies $R_{ij} = \lambda\, \bar{g}_{ij}$ (Einstein condition), then $\bar{g}$ is a fixed point of the normalized Ricci flow and $|\mathrm{Ric}(\bar{g})|_{\bar{g}} = |\lambda|\sqrt{n}$, $|\nabla \mathrm{Ric}(\bar{g})|_{\bar{g}} = 0$. Inequality (11) gives:

$$d_{\bar{g}}(p(T), p(0)) \;\le\; 2C\, |\lambda| \sqrt{n} \cdot \mathrm{Vol}_{\bar{g}}(\Omega) \cdot T.$$

The displacement is proportional to the scalar curvature λ, the elapsed time T, and the volume of the body. A flat body ($\lambda = 0$) does not move, as expected.

Convergent flows. If the Ricci flow converges to a flat metric g_∞ (e.g., in dimension 2 on surfaces of genus ≥ 2 [5]), then $|\mathrm{Ric}(g(t))|_g \to 0$ as $t \to \infty$. Hamilton's estimates [4] imply $\int_0^\infty |\mathrm{Ric}(g(t))|_g\, dt < \infty$, so $\mathcal{D}_\Omega < \infty$ even for $T = \infty$. The total displacement $d_{\bar{g}}(p(\infty), p(0))$ is therefore finite, and the center of mass converges to a well-defined limit point.

Lower Displacement Bound

The upper bound $d_{\bar{g}}(p(T), p(0)) \le C \cdot \mathcal{D}_\Omega$ of Theorem 1 raises a natural question: is $\mathcal{D}_\Omega$ an *equivalent* measure of the displacement, or merely a one-sided bound?

The homogeneous conformal deformation $g(t) = e^{2f(t)}\bar{g}$ shows that a lower bound cannot be universal: here $\mathcal{D}_\Omega > 0$ but $p(t) \equiv p(0)$, since the deformation scales all distances uniformly and leaves the center of mass invariant. The obstacle is that a homogeneous deformation is *invisible from* p: it acts symmetrically in all directions, producing no net force on the center of mass.

We introduce a condition that rules out this degeneracy.

Definition 4 (Non-degenerate deformation)**.** *For* $q \in M$ *and a symmetric* $(0,2)$*-tensor* h *on* Ω*, define:*

$$Q_h(q) \;:=\; \int_\Omega \rho(x) \int_0^1 h\big(\gamma'_{q,x}(s),\, \gamma'_{q,x}(s)\big)\, ds\, dV_{\bar{g}}(x), \tag{12}$$

where $\gamma_{q,x}$ is the minimizing $\bar{g}$-geodesic from q to x. Note that $Q_h : M \to \mathbb{R}$ is a smooth scalar function, and $\nabla_q Q_h \in T_q M$ is its Riemannian gradient. The deformation family $g(t)$ is μ-non-degenerate *along $p(t)$ if there exists $\mu > 0$ such that for all $t \in [0, T]$:*

$$\left|\nabla_q Q_{\dot{g}(t)}(q)\right|_{\bar{g}}\Big|_{q=p(t)} \;\geq\; \mu \, \|\dot{g}(t)\|_{L^1(\Omega,\bar{g})}. \tag{13}$$

Here the left-hand side has units of $[\text{length}]^{-1}$ (gradient of a scalar) and the right-hand side has units of $[\mu] \cdot [\text{area}]$ (a norm integrated over Ω), so μ carries units $[\text{length}]^{-(n+1)}$ where $n = \dim M$. The condition is scale-invariant: if Ω is rescaled by $\lambda > 0$, both sides scale identically.

Remark 2. *Condition (13) measures the spatial asymmetry of $\dot{g}$ as seen from $p(t)$: it requires that the geodesic average of $\dot{g}$ varies in space near $p(t)$. For a homogeneous deformation $\dot{g} = \phi(t)\,\bar{g}$, the function $Q_{\dot{g}}(q)$ is constant in q by symmetry, so $\nabla_q Q_{\dot{g}} = 0$ and condition (13) fails, consistently with $p(t) \equiv p(0)$.*

Theorem 2 (Lower Displacement Bound)**.** *Under the assumptions of Definition 3, suppose further that the deformation is μ-non-degenerate along $p(t)$ in the sense of Definition 4. Then there exists $c = c(\Omega, \rho, \bar{g}, \mu) > 0$ such that:*

$$\boxed{d_{\bar{g}}(p(T),\, p(0)) \;\geq\; c \cdot \mathcal{D}_\Omega.} \tag{14}$$

Combined with Theorem 1, the deformation functional is an equivalent measure of the displacement:

$$c \cdot \mathcal{D}_\Omega \;\leq\; d_{\bar{g}}(p(T), p(0)) \;\leq\; C \cdot \mathcal{D}_\Omega. \tag{15}$$

Proof. The proof proceeds in four steps, mirroring the proof of Theorem 1.

Step 1 — Lower bound on $|d/dt[\mathrm{grad}_q F]|$. We compute $\frac{d}{dt}[\mathrm{grad}_q F]|_{q=p(t)}$ precisely. From the expression $\mathrm{grad}_q F = -2\int_\Omega \rho(x) \exp^{-1}_{q,g(t)}(x)\, dV_{g(t)}$, differentiating in t and evaluating at $q = p(t)$:

$$\frac{d}{dt}\big[\mathrm{grad}_q F\big]\Big|_{q=p(t)} = -2\int_\Omega \rho(x) \frac{d}{dt}\big[\exp^{-1}_{q,g(t)}(x)\big] dV_{\bar{g}} + O(\delta^2).$$

The variation of the log map $\exp_q^{-1}(x)$ under $g \mapsto g + h$ is given by [6, Chapter 1]:

$$\frac{d}{dt}\big[\exp^{-1}_{q,g(t)}(x)\big] = -\tfrac{1}{2}\nabla_q \int_0^1 h\big(\gamma'_{q,x}(s), \gamma'_{q,x}(s)\big)\, ds + R_{\mathrm{curv}}(q, x, h),$$

where the curvature correction arises from the Jacobi equation along $\gamma_{q,x}$ and satisfies, by standard Jacobi field estimates:

$$|R_{\mathrm{curv}}(q, x, h)|_{\bar{g}} \leq C_0\, \kappa\, r^2\, |h|_{\bar{g}}(x),$$

with C_0 a universal constant, $\kappa = \sup K_{\bar{g}}$, and $r = d_{\bar{g}}(q, x) \leq \mathrm{diam}(\Omega)$. Integrating over Ω:

$$\frac{d}{dt}\big[\mathrm{grad}_q F\big]\Big|_{q=p(t)} = \nabla_q Q_{\dot{g}(t)}(q)\big|_{q=p(t)} + R_{\mathrm{curv}},$$

where $|R_{\text{curv}}| \leq C_0\,\kappa\,r^2\,\|\dot{g}(t)\|_{L^1(\Omega,\bar{g})}$. Choosing r small enough so that $C_0\,\kappa\,r^2 \leq \mu/2$ (which is compatible with the convexity hypothesis since $r < \pi/(2\sqrt{\kappa})$), and using condition (13):

$$\left|\frac{d}{dt}\big[\mathrm{grad}_q F\big]\right|_{\bar{g}} \geq \mu\,\|\dot{g}(t)\|_{L^1(\Omega)} - \frac{\mu}{2}\,\|\dot{g}(t)\|_{L^1(\Omega)} = \frac{\mu}{2}\,\|\dot{g}(t)\|_{L^1(\Omega)}. \tag{16}$$

Step 2 — Upper bound on $\mathrm{Hess}_q F$. The Hessian of F at $p(t)$ is bounded above: $\mathrm{Hess}_q F(p(t),t) \leq \Lambda(\Omega,\rho,\bar{g})\cdot \mathrm{Id}$, where $\Lambda < \infty$ by compactness of Ω and boundedness of $g(t)$.

Step 3 — Lower bound on $|dp/dt|$. From equation (5) of the proof of Theorem 1:

$$\left|\frac{dp}{dt}\right|_{\bar{g}} = \left|\mathrm{Hess}_q F^{-1}\cdot\frac{d}{dt}[\mathrm{grad}_q F]\right|_{\bar{g}} \geq \frac{1}{\Lambda}\left|\frac{d}{dt}[\mathrm{grad}_q F]\right|_{\bar{g}} \geq \frac{\mu}{2\Lambda}\,\|\dot{g}(t)\|_{L^1(\Omega)}.$$

Step 4 — Integration. Integrating over $[0,T]$:

$$d_{\bar{g}}(p(T),p(0)) \geq \int_0^T \left|\frac{dp}{dt}\right|_{\bar{g}} dt \geq \frac{\mu}{2\Lambda}\int_0^T \|\dot{g}(t)\|_{L^1(\Omega)}\,dt \geq \frac{\mu}{2\Lambda}\,\mathcal{D}_\Omega,$$

where the last inequality uses $\mathcal{D}_\Omega \geq \int_0^T \|\dot{g}(t)\|_{L^1(\Omega)}\,dt$ (since $\mathcal{D}_\Omega$ includes the additional term $\|\nabla\dot{g}\|$). Setting $c := \mu/(2\Lambda)$ gives (14). □ □

Open Problems

1. Moving bodies. The present results assume Ω is a fixed domain. For a body that deforms simultaneously with the ambient space — a *non-rigid body* $\Omega(t)$ — the center of mass functional involves an additional term arising from the variation of the domain. Extending Theorems 1 and 2 to this setting would require a careful treatment of the boundary variation and is left for future work.

2. Lorentzian extension and connection to cosmology. The present theorem is purely Riemannian. Its extension to a Lorentzian spacetime proceeds naturally via the ADM decomposition [1]: a spacetime $(\mathcal{M},\mathbf{g})$ is foliated into spacelike hypersurfaces Σ_t with induced spatial metric $\gamma(t)$, lapse N, and shift β^i, giving the identity

$$\dot{\gamma}_{ij} = -2NK_{ij} + \nabla_i\beta_j + \nabla_j\beta_i,$$

where K_{ij} is the extrinsic curvature of Σ_t. On each Σ_t, the induced metric $\gamma(t)$ is Riemannian, so the Karcher center of mass $p(t)$ is well-defined and Theorems 1 and 2 apply directly, with $\mathcal{D}_\Omega$ computed from $\dot{\gamma}$.

For a FLRW spacetime ($N=1$, $\beta=0$, $K_{ij} = -H\gamma_{ij}$):

$$\dot{\gamma}_{ij} = 2H(t)\,\gamma_{ij}, \qquad \mathcal{D}_\Omega[\mathrm{FLRW}] = \frac{2\sqrt{3}}{3}\,\mathrm{Vol}_{\gamma_0}(\Omega)\cdot[a(T)^3 - 1].$$

Via the Friedmann equation $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda)$, the functional $\mathcal{D}_\Omega$ captures contributions from both matter (ρ_m) and the cosmological constant (ρ_Λ). Whether the surplus of $\mathcal{D}_\Omega$ not attributable to matter provides a geometric interpretation of dark energy as the intrinsic motion of space is an open question that we leave for future investigation.

Conclusion

We have established that an extended body at rest is displaced by the sole deformation of the ambient space, and that this displacement is controlled quantitatively by the functional $\mathcal{D}_\Omega$. The lower bound (Theorem 2) shows that, under a natural non-degeneracy condition, $\mathcal{D}_\Omega$ is not merely a bound but an *equivalent measure* of the displacement:

$$c \cdot \mathcal{D}_\Omega \;\leq\; d_{\bar{g}}(p(T), p(0)) \;\leq\; C \cdot \mathcal{D}_\Omega.$$

The result — the Geometric Displacement Theorem and its converse — is a strict extension of Newton's law of inertia to the Riemannian setting: rest is possible if and only if the space is locally static, and the deformation functional measures exactly how much the space is responsible for the motion.

— Judicael Brindel, March 2026

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